Multicolor Ramsey Numbers for Complete Bipartite Versus Complete Graphs

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January 11, 2012

Abstract

Let $H_1, \dots, H_k$ be graphs. The multicolor Ramsey number $r(H_1, \dots, H_k)$ is the minimum integer r such that in every edge-coloring of K_r by k colors, there is a monochromatic copy of H_i in color i for some $1 \leq i \leq k$. In this paper, we investigate the multicolor Ramsey number $r(K_{2,t}, \dots, K_{2,t}, K_m)$, determining the asymptotic behavior up to a polylogarithmic factor for almost all ranges of t and m. Several different constructions are used for the lower bounds, including the random graph and explicit graphs built from finite fields. A technique of Alon and Rödl using the probabilistic method and spectral arguments is employed to supply tight lower bounds. A sample result is

$$c_1 \frac{m^2 t}{\log^4(mt)} \leq r(K_{2,t}, K_{2,t}, K_m) \leq c_2 \frac{m^2 t}{\log^2 m}$$

for some constants c_1 and c_2.

1 Introduction

The multicolor Ramsey number $r(H_1, \dots, H_k)$ is the minimum integer r such that in every edge-coloring of K_r by k colors, there is a monochromatic copy of H_i in color i for some $1 \leq i \leq k$. Ramsey's famous theorem [18] states that $r(K_s, K_t) < \infty$ for all s and t. Determining these numbers is usually a very difficult problem. Even determining the asymptotic behavior is difficult; there are only a few infinite families of graphs where the order of magnitude is known. A famous example is $r(K_3, K_m) = \Theta(m^2/\log m)$, where the upper bound was proved by Ajtai, Komlós, and Szemerédi [1] and the lower bound by Kim [13].

For more colors, in 1980 Erdős and Sós [9] conjectured that $r(K_3, K_3, K_m)/r(K_3, K_m) \to \infty$ as $m \to \infty$. This conjecture was open for 25 years until it was proved true by Alon and

*Research supported in part by NSF Grant 0969092.

H	$k=2$	$k\geq 3$
K_3	$\frac{m^3}{\log^{4+\delta} m} \ll r_2(K_3;K_m) \ll \frac{m^3 \log\log m}{\log^2 m}$	$\frac{m^{k+1}}{\log^{2k+\delta} m} \ll r_k(K_3;K_m) \ll \frac{m^{k+1}(\log\log m)^{k-1}}{\log^k m}$
C_4	$\frac{m^2}{\log^4 m} \ll r_2(C_4;K_m) \ll \frac{m^2}{\log^2 m}$	$r_k(C_4;K_m) = \Theta\left(\frac{m^2}{\log^2 m}\right)$
C_6	$\frac{m^{3/2}}{\log^3 m} \ll r_2(C_6;K_m) \ll \frac{m^{3/2}}{\log^{3/2} m}$	$r_k(C_6;K_m) = \Theta\left(\frac{m^{3/2}}{\log^{3/2} m}\right)$
C_{10}	$\frac{m^{5/4}}{\log^{5/2} m} \ll r_2(C_{10};K_m) \ll \frac{m^{5/4}}{\log^{5/4} m}$	$r_k(C_{10};K_m) = \Theta\left(\frac{m^{5/4}}{\log^{5/4} m}\right)$
$K_{s,t}$	$\frac{m^s}{\log^{2s} m} \ll r_2(K_{s,t};K_m) \ll \frac{m^s}{\log^s m}$	$r_k(K_{s,t};K_m) = \Theta\left(\frac{m^s}{\log^s m}\right)$

Table 1: Results on $r_k(H;K_m)$ proved by Alon and Rödl [2].

Rödl [2]. In their paper, they provided a general technique using graph eigenvalues and the probabilistic method which provides good estimates on multicolor Ramsey numbers. This breakthrough provided the first sharp asymptotic (up to a poly-log factor) bounds on infinite families of multicolor Ramsey numbers with at least three colors.

The exact results proved by Alon and Rödl [2] are shown in Table 1. For $k \geq 1$, define $r_k(H;G)$ to be $r(H,\ldots,H,G)$, where H is repeated k times. In other words, $r_k(H;G)$ is the minimum integer r such that in every edge-coloring of K_r by $k+1$ colors, there is a monochromatic copy of H in one of the first k colors or a copy of G in the $k+1$st color. In Table 1, s and t are fixed with $t \geq (s-1)!+1$, $\delta > 0$ is any positive constant, and m is going to infinity. Also, in the tables below, $a \ll b$ means there exists some positive constant c such that $a \leq cb$. All logarithms in this paper are base e.

One surprising aspect of Alon and Rödl's [2] techniques is that they prove very good upper and lower bounds for multicolor Ramsey numbers in cases where the two-color Ramsey number is not as well understood. For example, Erdős [8] conjectured that $r(C_4,K_m) = O(m^{2-\epsilon})$ for some absolute constant $\epsilon > 0$, and this conjecture is still open. The current best upper bound is an unpublished result of Szemerédi which was reproved by Caro, Rousseau, and Zhang [7] where they showed that $r(C_4,K_m) = O(m^2/\log^2 m)$ and the current best lower bound is $\Omega(m^{3/2}/\log m)$ by Bohman and Keevash [5]. In sharp contrast, for three colors Alon and Rödl [2] determined $r(C_4,C_4,K_m)$ up to a poly-log factor and found the order of magnitude of $r_k(C_4;K_m)$ for $k\geq 3$. A similar situation occurs for the other graphs in Table 1 besides K_3.

2 Results

We focus on the problem of determining $r_k(K_{2,t};K_m)$ when k is fixed and t is no longer a constant. Our results can be summarized by the following table; more precise statements are given later.

We are able to find the order of magnitude of $r_k(K_{2,t};K_m)$ up to a ploy-log factor for all

	$m \ll \log^2 t$	$\log^2 t \ll m \ll 2^t$	$2^t \ll m$
$k = 1$	$mt \ll r \ll \frac{m^2 t}{\log^2 m}$	$\frac{m^2 t}{\log^2 (mt)} \ll r \ll \frac{m^2 t}{\log^2 m}$	$r \ll \frac{m^2 t}{\log^2 m}$
$k = 2$	$mt \ll r \ll \frac{m^2 t}{\log^2 m}$	$\frac{m^2 t}{\log^2 (mt)} \ll r \ll \frac{m^2 t}{\log^2 m}$	$\frac{m^2 t}{\log^4 (mt)} \ll r \ll \frac{m^2 t}{\log^2 m}$
$k \geq 3$	$mt \ll r \ll \frac{m^2 t}{\log^2 m}$	$\frac{m^2 t}{\log^2 (mt)} \ll r \ll \frac{m^2 t}{\log^2 m}$	$\frac{m^2 t}{\log^2 (mt)} \ll r \ll \frac{m^2 t}{\log^2 m}$

Table 2: Results on $r = r_k(K_{2,t}; K_m)$ in this paper.

ranges of m and t except the upper right table cell where m is much larger than t and $k = 1$. This is similar to the fact that the order of magnitude of $r(C_4, K_m)$ is unknown but Alon and Rödl [2] found the order of magnitude up to a poly-log factor when $k \geq 2$. So the only remaining case is $r(K_{2,t}, K_m)$ when m is much larger than t. The best known lower bound is $r(K_{2,t}, K_m) \geq c_t (m/\log m)^{\rho(K_{2,t})}$, where $\rho(K_{2,t}) = 2 - \frac{2}{t}$ (see [3, 15].) Unfortunately, this lower bound has a constant c_t depending on t when we would like to know the exact order of magnitude.

The upper bound in Table 2 is a straightforward counting argument using the extremal number of $K_{2,t}$. Szemerédi (unpublished) and Caro, Rousseau, and Zhang [7] proved the following proposition for two colors; we extend it for all k using a related but slightly different technique.

Proposition 1. *For $k \geq 1$, $t \geq 2$, and $m \geq 3$ integers, there exists a constant c depending only on k such that*

$$r_k(K_{2,t}; K_m) \leq c \frac{m^2 t}{\log^2 m}.$$

The main contribution in this paper is the various lower bounds given in the table. One simple lower bound is to take $m-1$ vertex sets $X_1, \ldots, X_{m-1}$, each of size $t+1$. Color edges inside each X_i with one color and color all edges between X_is in the other color. This proves $r(K_{2,t}, K_m) > (m-1)(t+1)$. In fact, this proves the following proposition.

Proposition 2. *Let $k \geq 1$, $t \geq 2$, and $m \geq 3$ be integers. Then $r_k(K_{2,t}; K_m) > (m-1)(t+1)$.*

Note that being slightly more clever for $k \geq 2$ and making each X_i of size $r_k(K_{2,t}) - 1$ does not give a large improvement. A theorem of Lazebnik and Mubayi [16] proves that $r_k(K_{2,t}) > k^2(t-1)$ when k and t are prime powers and $r_k(K_{2,t}) \leq k^2(t-1) + k + 2$ for all k and t. Therefore the size of each X_i could be increased to roughly $k^2 t$ but that implies only a constant improvement in Proposition 2.

Another lower bound comes from the random graph $G(n,p)$. Consider a coloring of $E(K_n)$ obtained by taking k random graphs $G(n,p)$ as the first k colors and letting the last color be the remaining edges. Depending on the choice of n and p, this construction avoids $K_{2,t}$ in the first k colors and K_m in the last color. In Proposition 3, we show that

when $\log^2 t \ll m \ll 2^t$ it is possible to choose p so that $G(m^2t/\log^2(mt), p)$ avoids $K_{2,t}$ and has independence number at most m. When $m \gg 2^t$, the number of vertices must be reduced to roughly $m^{2-2/t}$ which does not provide a good lower bound on the Ramsey number. Most likely, when $m \ll \log^2 t$ a more detailed analysis shows that one can choose p so that $G(m^2t/\log^2(mt), p)$ avoids $K_{2,t}$ and has independence number at most m. We skip this analysis and only investigate Proposition 3 for $m \gg \log^2 t$ because when $m \ll \log^2 t$, the lower bound of mt from Proposition 2 is better than $m^2t/\log^2(mt)$. The precise statement of this lower bound is given in the following proposition.

Proposition 3. *Let $k \geq 1$, $t \geq 2$. For all constants $c_1, c_2 > 0$, there exists a constant $d > 0$ depending only on k and c_1, c_2 such that if $c_1 \log^2 t \leq m \leq c_2 2^t$ then $r_k(K_{2,t}; K_m) \geq d\frac{m^2t}{\log^2(mt)}$.*

Proposition 2 and Proposition 3 take care of the left two columns in Table 2. Proposition 2 works in both columns and most likely Proposition 3 also works in both columns, although we do not prove that since Proposition 2 is better when $m \ll \log^2 t$. What about the range $m \gg 2^t$? As mentioned, an extension of Proposition 3 using the random graph $G(n, p)$ gives a lower bound of $c_t m^{2-2/t}$ for some constant c_t depending on t. When t is constant, Alon and Rödl's [2] result from Table 1 shows lower bounds of $m^2/\log^4 m$ and $m^2/\log^2 m$ depending on k. If t is not fixed but still much smaller than m, we can prove the following precise lower bounds. This is our main theorem.

Theorem 4. *Let $t \geq 2$ and $k \geq 3$. There exists a constant $\rho > 0$ depending only on k such that the following holds.*

(i) If $m \geq 128 \log^2 t$, then $r(K_{2,t}, K_{2,t}, K_m) \geq \rho\frac{m^2t}{\log^4(mt)}$.

(ii) If $m \geq 16k \log t$, then $r_k(K_{2,t}; K_m) \geq \rho\frac{m^2t}{\log^2(mt)}$.

The construction in the above theorem works for $k \geq 2$ and (roughly) the rightmost two columns in Table 2. When $k = 2$, it is slightly worse than the random graph construction from Proposition 3 and matches it when $k \geq 3$. But it has the advantage over the random graph of working in the rightmost column of Table 2, where m is much larger than t. Also, the construction only works for $k \geq 2$, which is the reason for the missing lower bound in the upper right cell of Table 2.

This construction is an algebraic graph construction using finite fields and is similar to a construction by Lazebnik and Mubayi [16], which in turn was based on constructions of Axenovich, Füredi, and Mubayi [4] and Füredi [11]. A theorem of Alon and Rödl [2] which relates the second largest eigenvalue of a graph with the number of the independent sets is then used to show the construction is a good choice for a $K_{2,t}$-free graph with small independence number. The properties of the construction are stated in the following theorem.

Theorem 5. *For any prime power q and any integer $t \geq 2$ such that $q \equiv 0 \pmod t$ or $q \equiv 1 \pmod t$, there exists a graph G with the following properties:*

- *G has $q(q-1)/t$ vertices,*

- *G has no multiple edges but some vertices have loops,*
- *G is regular of degree $q-1$ (loops contribute one to the degree),*
- *G is $K_{2,t+1}$-free,*
- *the second largest eigenvalue of the adjacency matrix of G is $\sqrt{q}$.*

Several open problems remain: in Table 2 are the upper or lower bounds correct? The upper and lower bounds are very close; we are fighting against a poly-log term. But it would still be interesting to know which bounds are correct. One of the differences is a $\log^2 m$ versus a $\log^2(mt)$ in the denominator. If m is much larger than t then $\log^2 m \sim \log^2(mt)$, but in the left two columns the gap starts to widen. As m gets smaller relative to t, the $m^2t/\log^2(mt)$ lower bound eventually becomes worse than a really simple mt lower bound. In an effort to understand this gap between the upper and lower bounds, we focused on the special case of $r(K_{2,t}, K_3)$.

Theorem 6. *Let $t \geq 2$. Then*

$$2t \leq r(K_{2,t}, K_3) \leq (2.155 + o(1))t$$

A consequence of this theorem is that we partially verify the following conjecture of Sidorenko [19]: for all graphs H, $r(H, K_3) \leq |V(H)| + |E(H)|$. Theorem 6 shows Sidorenko's conjecture is true for $H = K_{2,t}$ (there exist other easier proofs which show $r(K_{2,t}, K_3) \leq 3t+1$.)

Other open problems include $r(K_{2,t}, K_m)$ when m is much larger than t and $r_k(K_{s,t}; K_m)$ when s is larger than two. Using ideas from the projective norm graphs, the construction in Section 4 can be extended to use norms to forbid $K_{s,t}$ for s fixed, at the expense of more complexity in the proof of the spectrum. Thus the remaining problem on $r_k(K_{s,t}; K_m)$ is to investigate when s, t, and m are all going to infinity. In other words, how do the constants (implicit) in Table 2 depend on s? Comments about these and other open problems are discussed in Section 6.

3 The Ramsey Numbers $r_k(K_{2,t}; K_m)$

In this section we prove all the upper and lower bounds given in Table 2: Proposition 1 in Section 3.1, Proposition 3 in Section 3.2, and Theorem 4 in Section 3.3.

3.1 An upper bound

In this section, we prove Proposition 1. For two colors, the proposition was first proved in the 1980s by Szemerédi but he never published a proof. Caro, Rousseau, and Zhang [7] published a proof in 2000 and Jiang and Salerno [12] gave another more general proof but still for two colors. We use a slightly different (but closely related) proof technique inspired by Alon and Rödl [2] to extend the upper bound to three or more colors. First, we need the following two theorems. If F is a graph and n is an integer, define $ex(n, F)$ to be the maximum number of edges in an n-vertex graph which does not contain F as a subgraph.

Theorem 7. *(Kövari, Sós, Turán [14]) For $2 \le t \le n$, $ex(n, K_{2,t}) \le \frac{1}{2}\sqrt{t-1}n^{3/2} + \frac{n}{2} \le \sqrt{t}n^{3/2}$.*

The following theorem is a corollary of the famous result of Ajtai, Komlós, and Szemerédi [1] on $r(K_3, K_m)$ (see also [6].)

Theorem 8. *There exists an absolute constant c such that the following holds. Let G be an n-vertex graph with maximum degree d and let s be the number of triangles in G. Then*

$$\alpha(G) \ge \frac{cn}{d} \log\left(\frac{nd^2}{s}\right).$$

We will apply this theorem in a graph where we can bound the average degree and know a bound on the number of edges in any neighborhood; using standard tricks the theorem can be changed to use average degree.

Corollary 9. *There exists an absolute constant c such that the following holds. Let G be an n-vertex graph with average degree at most d, where for every vertex $v \in V(G)$, every $2d$-subset of $N(v)$ spans at most d^2/f edges. Then the independence number of G is at least $\frac{cn \log f}{d}$.*

Proof. Let H be the subgraph of G formed by deleting all vertices with degree bigger than $2d$. H has at least half the vertices of G since G has average degree at most d; in addition H has maximum degree $2d$. Also, H has at most $s = nd^2/f$ triangles since each neighborhood of a vertex in H spans at most d^2/f edges. Thus Theorem 8 implies there exists a constant c so that

$$\alpha(G) \ge \frac{cn}{2d} \log\left(\frac{n(2d)^2}{s}\right) = \frac{cn}{2d} \log(4f) \ge \frac{cn}{2d} \log f.$$

$\square$

Proof of Proposition 1. Let c_1 be the constant from Corollary 9; note that we can assume $c_1 \le 1$. Define $c_2 = \frac{256k^2}{c_1^2}$ and assume $n > \frac{c_2 m^2 t}{\log^2 m}$. Consider a $(k+1)$-coloring of $E(K_n)$ and let C_i be the graph whose edges are the ith color class for $i = 1, \ldots, k$. Assume C_i is $K_{2,t}$-free for all $1 \le i \le k$. We will show that the independence number of $C_1 \cup \cdots \cup C_k$ is at least m, which will imply the $(k+1)$-st color class contains a copy of K_m; i.e. $r_k(K_{2,t}; K_m) \le \frac{c_2 m^2 t}{\log^2 m}$.

Since $C_1, \ldots, C_k$ are $K_{2,t}$-free, they each have at most $\sqrt{t}n^{3/2}$ edges by Theorem 7. Let $G = C_1 \cup \cdots \cup C_k$ so $|E(G)| \le k\sqrt{t}n^{3/2}$. Let $d = 2k\sqrt{tn}$, so that G has average degree at most d. Consider some vertex $v \in V(G)$ and let $A \subseteq N(v)$ with $|A| = 2d$. Then $C_i[A]$ is $K_{2,t}$-free for $1 \le i \le k$ so $|E(G[A])| \le k \cdot ex(2d, K_{2,t}) \le 4k\sqrt{t}d^{3/2}$. To apply Corollary 9, we need to solve the following for f:

$$4k\sqrt{t}d^{3/2} = \frac{d^2}{f}.$$

The solution is $f = \frac{1}{4k}\sqrt{d/t}$ so Corollary 9 implies G contains an independent set of size $\frac{c_1 n \log f}{d}$. To complete the proof, we just need to show this is at least m. Use the definitions of $d = 2k\sqrt{tn}$ and $f = \frac{1}{4k}\sqrt{d/t}$ to obtain

$$\alpha(G) \geq \frac{c_1 n}{d} \log f = \frac{c_1 n}{2k\sqrt{tn}} \log\left(\frac{1}{4k}\frac{\sqrt{2k}\sqrt[4]{tn}}{\sqrt{t}}\right) = \frac{c_1}{2k}\sqrt{\frac{n}{t}} \log\left(\frac{1}{2\sqrt{2k}}\sqrt[4]{\frac{n}{t}}\right).$$

Recall that we assumed $n > \frac{c_2 m^2 t}{\log^2 m}$, so

$$\alpha(G) \geq \frac{c_1}{2k}\sqrt{\frac{c_2 m^2}{\log^2 m}} \log\left(\frac{1}{2\sqrt{2k}}\sqrt[4]{\frac{c_2 m^2}{\log^2 m}}\right).$$

Use that $c_2 = \frac{256k^2}{c_1^2}$ and simplify to obtain

$$\alpha(G) \geq \frac{8m}{\log m} \log\left(\sqrt{\frac{\sqrt{c_2}}{8k} \cdot \frac{m}{\log m}}\right) = \frac{4m}{\log m} \log\left(\frac{2}{c_1}\frac{m}{\log m}\right).$$

Since $c_1 \leq 1$,

$$\alpha(G) \geq \frac{4m}{\log m} \log\left(\frac{m}{\log m}\right) = \frac{4m}{\log m}\left(\log m - \log\log m\right) \geq m.$$

The last inequality uses $\log m \geq \frac{4}{3}\log\log m$ which is true for $m \geq 3$. $\square$

3.2 The Random Graph

In this section, we prove Proposition 3 by using the random graph $G(n,p)$.

Lemma 10. *For all constants c_1, c_2, there exists a constant c_3 such that the following holds. Given two integers t and m with $c_1 \log^2 t \leq m \leq c_2 2^t$, let $n = c_3 \frac{m^2 t}{\log^2(mt)}$ and $p = \sqrt{\frac{t}{e^8 n}}$. Then with probability tending to 1 as m tends to infinity ($m \to \infty$ implies $t, n \to \infty$ as well), $G(n,p)$ is $K_{2,t}$-free and has independence number at most m.*

Proof. Let $c_3 = \min\{\frac{1}{c_2^2}, \frac{1}{400e^8}\}$. The expected number of $K_{2,t}$s is upper bounded by

$$n^2 \binom{n}{t} p^{2t} \leq n^2 \left(\frac{en}{t}\right)^t \left(\frac{t}{e^8 n}\right)^t = n^2 e^{-7t}. \tag{1}$$

We want this to go to zero as $m \to \infty$, so it suffices to show that t is bigger than roughly $\log n$. Using the definition of n, upper bound $\log n$ by

$$\log n = \log\left(c_3 \frac{m^2 t}{\log^2(mt)}\right) \leq 2\log m + \log t + \log c_3$$

But since $m \leq c_2 2^t \leq c_2 e^t$,

$$\log n \leq 2(\log c_2 + t) + \log t + \log c_3 \leq 2t + \log t + 2\log c_2 + \log c_3.$$

Since $c_3 \leq \frac{1}{c_2^2}$, $2\log c_2 + \log c_3 \leq 0$. Using that $\log t \leq t$, we obtain $\log n \leq 3t$, which when combined with (1) shows the expected number of $K_{2,t}s$ is upper bounded by

$$n^2 e^{-7t} = e^{2\log n - 7t} \leq e^{-t}.$$

Since $m \to \infty$ implies $t \to \infty$, the expected number of $K_{2,t}$s goes to zero as $m \to \infty$.

Let $d = pn$. When $d = o(n)$, the independence number of $G(n,p)$ is concentrated around $\frac{2n}{d}\log d$. More precisely, Frieze [10] (see also [3, 6]) proved that for fixed $\epsilon > 0$ and $d = o(n)$, with probability going to one as $n \to \infty$, the independence number of $G(n,p)$ is within $\frac{\epsilon n}{d}$ of $\frac{2n}{d}(\log d - \log\log d - \log 2 + 1)$. First, note that since $c_1 \log^2 t \leq m$, $m^2 t/\log^2(mt) \to \infty$ as $m \to \infty$. This implies $n/t \to \infty$ which implies $d = pn = o(n)$, so the result of Frieze [10] can be applied. Therefore, w.h.p.

$$\alpha(G(n,p)) < 10\frac{2n}{pn}\log(pn) = 20e^4\sqrt{\frac{n}{t}}\log\left(\sqrt{\frac{nt}{e^8}}\right) \leq 10e^4\sqrt{\frac{n}{t}}\log(nt).$$

The next step is to show that when the definition of n is inserted, the expression is at most m showing w.h.p. the independence number of $G(n,p)$ is at most m. The computations are very similar to the end of the proof of Proposition 1 in Section 3.1.

$$\alpha(G(n,p)) < 10e^4\sqrt{c_3}\frac{m}{\log(mt)}\log\left(\frac{c_3 m^2 t^2}{\log^2(mt)}\right) \leq 20e^4\sqrt{c_3}m \leq m.$$

Therefore, as m tends to infinity, the probability that $G(n,p)$ contains a copy of $K_{2,t}$ or has independence number at least m tends to zero, completing the proof. □

Proof of Proposition 3. Color $E(K_n)$ by taking k copies of $G(n,p)$ (with $p = \sqrt{t/(e^8 n)}$) as the first k colors and letting the $k+1$st color be the remaining edges. Lemma 10 shows w.h.p. the first k colors are $K_{2,t}$-free (since k is fixed.) A single $G(n,p)$ has w.h.p. independence number at most m so the clique number of the $k+1$st color is certainly at most m. □

3.3 An algebraic lower bound

In this subsection, we prove Theorem 4. Our main tool is the following very general theorem from Alon and Rödl [2]. Their idea is to take an H-free graph G and construct k graphs $G_1, \ldots, G_k$ by taking k random copies of G. In other words, fix some set W of size $|V(G)|$ and let G_i be the graph obtained by a random bijection between $V(G)$ and W. We now have a $k+1$ coloring of the edges of the complete graph on vertex set W: let the first k colors be $G_1, \ldots, G_k$ and let the $k+1$st color be the edges outside any G_i. Alon and Rödl's key insight is that if we know the second largest eigenvalue of G, then G is an expander graph which implies some knowledge about the independent sets in G. This is then used to bound the independence number of $G_1 \cup \cdots \cup G_k$, in other words obtain an estimate of m.

Theorem 11. *(Alon and Rödl, Theorem 2.1 and Lemma 3.1 from [2]) Let G be an n-vertex, H-free, d-regular graph where G has no multiple edges but some vertices have loops and let $k \geq 2$ be any integer. Let λ be the second largest eigenvalue in absolute value of the adjacency matrix of G. If $m \geq \frac{2n}{d}\log n$ and*

$$\left(\frac{emd^2}{4\lambda n \log n}\right)^{\frac{2kn\log n}{d}} \left(\frac{2e\lambda n}{md}\right)^{km} \left(\frac{m}{n}\right)^{m(k-1)} < 1 \tag{2}$$

then $r_k(H; K_m) > n$.

A combination of Theorem 11 and Theorem 5 plus the density of the prime numbers proves Theorem 4. To be able to apply Theorem 5, we need to find a prime power q which is congruent to zero or one modulo t and is in the required range. Recall that we are targeting a bound of $\frac{m^2 t}{\log^4(mt)}$ or $\frac{m^2 t}{\log^2(mt)}$ and the number of vertices from Theorem 5 is $q(q-1)/t$. Given inputs m and t, we therefore want to find a prime power q so that $q \equiv 0 \pmod t$ or $q \equiv 1 \pmod t$ and $q(q-1)/t$ is near $\frac{m^2 t}{\log^{2s}(mt)}$ where s is one or two. This can be accomplished using the Prime Number Theorem.

Lemma 12. *Fix integers $s, L \geq 1$. There exists a constant $\delta > 0$ depending only on s and L such that the following holds. For every $t \geq 2$ and $m \geq 4^s L \log^s t$, either $\frac{\delta m^2 t}{L^2 \log^{2s}(mt)} \leq 2$ or there is a prime power q so that $q \equiv 1 \pmod t$ and*

$$\delta \frac{m^2 t}{L^2 \log^{2s}(mt)} \leq \frac{q(q-1)}{t} \leq \frac{m^2 t}{L^2 \log^{2s}(mt)}.$$

The proof of this lemma is given in Appendix A. Now a combination of Lemma 12, Theorem 5, and Theorem 11 plus some computations proves Theorem 4 *(i)*.

Proof of Theorem 4 (i). Suppose $t \geq 2$, $k = 2$, and $m \geq 128 \log^2 t$ are given. Fix $s = 2$ and $L = 8$ so that the conditions of Lemma 12 are satisfied. Choose q and δ according to Lemma 12. Note that if $\frac{\delta m^2 t}{L^2 \log^4(mt)} \leq 2$, then trivially $r(K_{2,t}, K_{2,t}, K_m) \geq 2 \geq \frac{\delta m^2 t}{L^2 \log^4(mt)}$. Therefore, assume that

$$\delta \frac{m^2 t}{64 \log^4(mt)} \leq \frac{q(q-1)}{t} \leq \frac{m^2 t}{64 \log^4(mt)}. \tag{3}$$

Let G be the graph from Theorem 5. Then d (the average degree) is $q-1$, λ (the second largest eigenvalue in absolute value) is $\sqrt{q}$, and $n = q(q-1)/t$.

To apply Theorem 11, we need to show that $m \geq \frac{2n}{d}\log n$ and also show k, m, λ, n, and d satisfy the inequality (2). We break this into two steps: first we show that $m \geq \frac{n}{d}\log^2 n \geq \frac{2n}{d}\log n$ using the choice of q from Lemma 12. Next, we let $m' = \frac{n}{d}\log^2 n$ and check the inequality (2) with k, m', λ, n, and d. This shows $r_k(K_{2,t}; K_{m'}) > n$, and since $m \geq m'$, this implies $r_k(K_{2,t}; K_m) > n$. Using that $n = q(q-1)/t$, equation (3) shows $n > \frac{1}{64}\delta m^2 t / \log^4(mt)$. If $\rho \leq \frac{\delta}{64}$, we have proved $r(K_{2,t}, K_{2,t}, K_m) \geq \rho m^2 t / \log^4(mt)$. Also,

note that we can assume $n > n_0$ for some constant n_0 by choosing $\rho = \frac{\delta}{64n_0}$ (since then $n \leq n_0$ implies $\rho m^2 t/\log^4(mt) \leq 1$.)

Step 1 We want to show $m \geq \frac{n}{d}\log^2 n$. Start with (3):

$$\begin{aligned} n &\leq \frac{m^2 t}{64\log^4(mt)} \\ 64n\log^4(mt) &\leq m^2 t. \end{aligned} \tag{4}$$

Take the log of both sides, to obtain

$$\begin{aligned} \log 64 + \log n + \log\log^4(mt) &\leq 2\log m + \log t \leq 2\log(mt) \\ \log n &\leq 2\log(mt). \end{aligned}$$

Combining this with (4) yields

$$\begin{aligned} & n\log^4 n \leq 16n\log^4(mt) \leq \frac{1}{4}m^2 t \\ \Rightarrow \quad & m^2 \geq \frac{4n}{t}\log^4 n \\ \Rightarrow \quad & m \geq 2\sqrt{\frac{n}{t}}\log^2 n = \frac{2\sqrt{q(q-1)}}{t}\log^2 n \geq \frac{q}{t}\log^2 n = \frac{n}{d}\log^2 n. \end{aligned}$$

Step 2 Let $m' = \frac{n}{d}\log^2 n$. We need to verify that

$$\left(\frac{em'd^2}{4\lambda n\log n}\right)^{\frac{2kn\log n}{d}} \left(\frac{2e\lambda n}{m'd}\right)^{km'} \left(\frac{m'}{n}\right)^{m'(k-1)} < 1.$$

Substitute in $k = 2$ and $m' = \frac{n}{d}\log^2 n$ in the exponent of the LHS and then take the m'th-root to obtain

$$\Lambda := \left(\frac{em'd^2}{4\lambda n\log n}\right)^{\frac{4}{\log n}} \left(\frac{2e\lambda n}{m'd}\right)^2 \left(\frac{m'}{n}\right).$$

We must show $\Lambda < 1$. Substitute in $m' = \frac{n}{d}\log^2 n$ and simplify to obtain

$$\Lambda = \left(\frac{ed\log n}{4\lambda}\right)^{\frac{4}{\log n}} \left(\frac{4e^2\lambda^2}{\log^4 n}\right)\left(\frac{\log^2 n}{d}\right) = \left(\frac{e^4 d^4\log^4 n}{256\lambda^4}\right)^{\frac{1}{\log n}} \left(\frac{4e^2\lambda^2}{d\log^2 n}\right). \tag{5}$$

Now $d = q-1$, $\lambda = \sqrt{q}$, and $n = q(q-1)/t$ so $\lambda^2/d = q/(q-1) \leq 2$ and

$$\frac{d^4}{\lambda^4} = \frac{(q-1)^4}{q^2} < q(q-1) = nt < n^2.$$

Insert these inequalities into (5) to obtain

$$\Lambda < \left(\frac{e^4 n^2 \log^4 n}{256}\right)^{\frac{1}{\log n}} \left(\frac{8e^2}{\log^2 n}\right).$$

Since $n^2 = e^{2\log n}$ raised to the power $1/\log n$ is a constant, when n gets big the above expression drops below 1 (as mentioned above, we can assume $n > n_0$.) Therefore, Theorem 11 implies that $r(K_{2,t}, K_{2,t}, K_{m'}) > n$. In Step 1, we showed that $m \geq m'$ so $r(K_{2,t}, K_{2,t}, K_m) > n$. Since $n = q(q-1)/t$, equation (3) shows that $n > \frac{1}{64}\delta\frac{m^2 t}{\log^4 mt}$, completing the proof. □

Proof sketch of Theorem 4 (ii). Given m, t, and $k \geq 3$, fix $s = 1$ (instead of 2) and $L = 4k$ and choose q and δ according to Lemma 12. The proof is mostly the same as the above proof, except we choose $m' = 2k\frac{n}{d}\log n$ (the difference is that the log is not squared plus now there is a $2k$ out front.) The proof then proceeds in two steps: show that $m \geq 2k\frac{n}{d}\log n = m'$ and then show that k, m', λ, n, and d satisfy the inequality (2). Showing $m \geq m'$ is almost identical to Step 1 in the previous proof. Showing k, m', λ, n, and d satisfy inequality (2) in Theorem 11 is tedious; the details are in Appendix B. □

4 An algebraic $K_{2,t+1}$-free construction

To prove Theorem 5, we construct two different graphs for the two cases: one graph G^+ for $q \equiv 0 \pmod t$ and one graph $G^\times$ for $q \equiv 1 \pmod t$. The two graphs are closely related; they are built from finite fields. Fix a prime p and an integer a, and let $q = p^a$. Let $\mathbb{F}_q$ be the finite field of order q and let $\mathbb{F}_q^*$ be the finite field of order q without the zero element.

When $q \equiv 0 \pmod t$, let H be an additive subgroup of $\mathbb{F}_q$ of order t. Such a subgroup exists since t divides q so $t = p^b$ for some $b \leq a$. Define a graph G^+ as follows. Let $V(G^+) = (\mathbb{F}_q/H) \times \mathbb{F}_q^*$. We will write elements of $\mathbb{F}_q/H$ as $\bar{a}$, where $\bar{a}$ as the additive coset of H generated by a. That is, $\bar{a} = \{h + a : h \in H\}$. For $\bar{a}, \bar{b} \in \mathbb{F}_q/H$ and $x, y \in \mathbb{F}_q^*$, make $(\bar{a}, x)$ adjacent to $(\bar{b}, y)$ if $xy \in \overline{a+b}$. Since H is a normal subgroup the coset $\overline{a+b}$ is well-defined, so by $xy \in \overline{a+b}$ we mean there exists some $h \in H$ such that $xy = h + a + b$. [1]

When $q \equiv 1 \pmod t$, let H be a multiplicative subgroup of $\mathbb{F}_q^*$ of order t. Such a subgroup exists since t divides the order of $\mathbb{F}_q^*$ and $\mathbb{F}_q^*$ is a cyclic multiplicative group. Define a graph $G^\times$ as follows. Let $V(G^\times) = \left(\mathbb{F}_q^*/H\right) \times \mathbb{F}_q$. For $\bar{a}, \bar{b} \in \mathbb{F}_q^*/H$ and $x, y \in \mathbb{F}_q$, make $(\bar{a}, x)$ adjacent to $(\bar{b}, y)$ if $x + y \in \overline{ab}$. [2]

4.1 Simple properties of G^+ and $G^\times$

Lemma 13. *G^+ and $G^\times$ are regular of degree $q-1$.*

[1] In finite fields, additive subgroups of a given order are isomorphic as groups. Each element of $\mathbb{F}_q$ has additive order the characteristic, so H decomposes into p^{b-1} orbits of size p and one can obtain a group isomorphism by mapping orbits to orbits. Therefore, G^+ is uniquely defined up to isomorphism.

[2] In finite fields, multiplicative subgroups of a given order are isomorphic as groups since $\mathbb{F}_q^*$ is cyclic. Therefore, $G^\times$ is uniquely defined up to isomorphism.

Proof. First, consider G^+. Fix some vertex $(\bar{a}, x) \in V(G^+)$ and pick $y \in \mathbb{F}_q^*$ ($q-1$ choices.) The element xy is now in some coset $\bar{c}$. Since the cosets form a group, the coset $\overline{c-a}$ is well-defined. Thus $(\bar{a}, x)$ is adjacent to $(\overline{d}, y)$ in G^+ if and only if $\overline{d} = \overline{xy-a}$.

Now consider $G^\times$. Fix some vertex $(\bar{a}, x) \in V(G^\times)$ and pick $y \in \mathbb{F}_q$. If $x \neq -y$, then there is a coset $\bar{c}$ containing $x+y$. Since the cosets form a group, the coset $\overline{ca^{-1}}$ is well defined. If $x = -y$, then there is no coset which contains zero. Thus $(\bar{a}, x)$ is adjacent to $(\overline{d}, y)$ if and only if $x \neq -y$ and $\overline{d} = \overline{(x+y)a^{-1}}$. Therefore $(\bar{a}, x)$ is adjacent to $q-1$ vertices, since there are $q-1$ choices for $y \in \mathbb{F}_q$ with $x \neq -y$. □

Lemma 14. *The common neighborhood of any two vertices in G^+ has size exactly t.*

Proof. The proof is similar to the proofs given in [11, 16]. Fix $\bar{a}, \bar{b} \in \mathbb{F}_q/H$ and $x, y \in \mathbb{F}_q^*$ and consider the common neighborhood of the vertices $(\bar{a}, x)$ and $(\bar{b}, y)$. A vertex $(\bar{c}, z)$ will be adjacent to both of $(\bar{a}, x)$ and $(\bar{b}, y)$ if

$$\begin{aligned} xz &\in \overline{a+c} \\ yz &\in \overline{b+c}. \end{aligned}$$

In other words, there exists some $h_1, h_2 \in H$ such that

$$\begin{aligned} xz &= a + c + h_1 \\ yz &= b + c + h_2. \end{aligned}$$

So fix $h_1, h_2 \in H$ and count how many choices there are for c and z so that $(\bar{c}, z)$ is adjacent to both $(\bar{a}, x)$ and $(\bar{b}, y)$ using h_1 and h_2. We show there is a unique c and z. Say we had c, c', z, z' such that

$$xz = a + c + h_1 \tag{6}$$
$$yz = b + c + h_2 \tag{7}$$
$$xz' = a + c' + h_1 \tag{8}$$
$$yz' = b + c' + h_2. \tag{9}$$

Add (6) to (9); this equals (7) plus (8).

$$\begin{aligned} xz + yz' &= a + b + c + c' + h_1 + h_2 = yz + xz' \\ (x-y)(z-z') &= 0. \end{aligned} \tag{10}$$

If $x = y$, then subtracting (6) from (7) gives $a - b \in H$ which means $\bar{a} = \bar{b}$. But now $(\bar{a}, x)$ and $(\bar{b}, y)$ are the same vertex. Thus (10) implies $z = z'$. Then subtracting (6) and (8) we get $c = c'$, showing there is a unique c, z such that $(\bar{c}, z)$ is adjacent to both $(\bar{a}, x)$ and $(\bar{b}, y)$ using h_1, h_2. (Note that not only is there a unique $(\bar{c}, z)$, but the choice of the representative c for the coset $\bar{c}$ is unique.)

There are now t^2 choices for h_1 and h_2 and each provides a unique c, z. But each coset $\bar{c}$ has t elements so there are exactly $t^2/t = t$ common neighbors of $(\bar{a}, x)$ and $(\bar{b}, y)$. □

Lemma 15. *The common neighborhood of any two vertices in $G^\times$ has size exactly t.*

Proof. Fix $\bar{a}, \bar{b} \in \mathbb{F}_q^*/H$ and $x, y \in \mathbb{F}_q$ and consider the common neighborhood of the vertices $(\bar{a}, x)$ and $(\bar{b}, y)$. A vertex $(\bar{c}, z)$ will be adjacent to both $(\bar{a}, x)$ and $(\bar{b}, y)$ if

$$\begin{aligned} x + z &\in \overline{ac} \\ y + z &\in \overline{bc}. \end{aligned}$$

In other words, there exists some $h_1, h_2 \in H$ such that

$$\begin{aligned} x + z &= h_1 ac \\ y + z &= h_2 bc. \end{aligned}$$

So fix some $h_1, h_2 \in H$ and count how many choices there are for c and z so that $(\bar{c}, z)$ is adjacent to both $(\bar{a}, x)$ and $(\bar{b}, y)$ using h_1 and h_2. We show there is a unique such c and z. Say there existed c, c', z, z' such that

$$\begin{aligned} x + z &= h_1 ac && (11) \\ y + z &= h_2 bc && (12) \\ x + z' &= h_1 ac' && (13) \\ y + z' &= h_2 bc'. && (14) \end{aligned}$$

Multiply (11) by (14), which equals (12) times (13).

$$\begin{aligned} (x + z)(y + z') &= h_1 h_2 abcc' = (y + z)(x + z') \\ xy + xz' + yz + zz' &= xy + yz' + xz + zz' \\ xz' + yz &= xz + yz' \\ (x - y)(z' - z) &= 0 && (15) \end{aligned}$$

If $x = y$, then (11) and (12) show

$$\begin{aligned} h_1 ac &= x + z = y + z = h_2 bc \\ ab^{-1} &= h_1^{-1} h_2 \in H \end{aligned}$$

which shows $\bar{a} = \bar{b}$. But now $(\bar{a}, x)$ and $(\bar{b}, y)$ are the same vertex. Thus (15) implies $z = z'$. But now (11) and (13) show $c = c'$.

Thus for every choice of $h_1, h_2 \in H$ there is a unique c, z such that $(\bar{c}, z)$ is adjacent to both $(\bar{a}, x)$ and $(\bar{b}, y)$ using h_1, h_2. Note that not only is there a unique $(\bar{c}, z)$, but the choice of the representative c for the coset $\bar{c}$ is unique. There are now t^2 choices for h_1 and h_2 and each provides a unique c, z. But each coset $\bar{c}$ has t elements so there are exactly $t^2/t = t$ common neighbors of $(\bar{a}, x)$ and $(\bar{b}, y)$. □

4.2 The Spectrum of G^+ and $G^\times$

Lemma 16. *The eigenvalues of G^+ are $q-1$, $\pm\sqrt{q}$, ± 1, and 0. If p is an odd prime, they have the following multiplicities: $q-1$ has multiplicity 1, $\sqrt{q}$ and $-\sqrt{q}$ each have multiplicity $\frac{1}{2}(q/t-1)(q-2)$, 1 and -1 both have multiplicity $\frac{1}{2}(q/t-1)$, and 0 has multiplicity $q-2$.*

Lemma 17. *The eigenvalues of $G^\times$ are $q-1$, $\pm\sqrt{q}$, ± 1, and 0. If p is an odd prime, they have the following multiplicities: $q-1$ has multiplicity 1, $\sqrt{q}$ and $-\sqrt{q}$ each have multiplicity $\frac{1}{2}((q-1)/t-1)(q-1)$, 1 and -1 both have multiplicity $\frac{1}{2}(q-1)$, and 0 has multiplicity $(q-1)/t-1$.*

The proof of these lemmas are similar to proofs by Alon and Rödl [2, Lemma 3.6] and Szabó [20]. In addition, the two proofs given below are almost the same but there are several subtle issues involving the fact that G^+ and $G^\times$ switch between $\mathbb{F}_q$ and $\mathbb{F}_q^*$. There are small but crucial differences in how the proofs below handle the zero element. Therefore, we give both proofs and caution the reader to pay attention to how the zero element is handled when reading the proofs.

Proof of Lemma 16. Let M be the adjacency matrix of G^+. Let χ be an arbitrary additive character of $\mathbb{F}_q/H$ and let ϕ be an arbitrary multiplicative character of $\mathbb{F}_q^*$. This means that

$$\chi : \mathbb{F}_q/H \to \mathbb{C} \qquad \phi : \mathbb{F}_q^* \to \mathbb{C}$$

where χ is an additive group homomorphism (if $\bar{a}, \bar{b}$ are cosets in $\mathbb{F}_q/H$ then $\chi(\bar{a}+\bar{b}) = \chi(\bar{a})\chi(\bar{b})$, $\chi(\bar{0}) = 1$, and $\chi(-\bar{a}) = \chi(\bar{a})^{-1}$) and ϕ is a multiplicative group homomorphism (if $a, b \in \mathbb{F}^*/q$ then $\phi(ab) = \phi(a)\phi(b)$, $\phi(1) = 1$, $\phi(a^{-1}) = \phi(a)^{-1}$.) Note that since $\phi(1) = 1$ and $x^q = 1$ for any $x \in \mathbb{F}_q^*$, $\phi(x)$ must be a root of unity in $\mathbb{C}$. Thus $\phi(x^{-1}) = \phi(x)^{-1} = \overline{\phi(x)}$ where $\overline{\phi(x)}$ is the complex conjugate of $\phi(x)$. Similarly, $\chi(-\bar{a}) = \overline{\chi(\bar{a})}$, the complex conjugate of χ applied to the coset $\bar{a}$.

Let $\langle \chi, \phi \rangle$ denote the column vector whose coordinates are labeled by the elements of $V(G^+)$ and whose entry at the coordinate $(\bar{a}, x)$ is $\chi(\bar{a})\phi(x)$. We now show that $\langle \chi, \phi \rangle$ is an eigenvector of M and compute its eigenvalue. The following expression is the entry of the vector $M\langle \chi, \phi \rangle$ at the coordinate $(\bar{a}, x)$.

$$\sum_{\substack{(\bar{b},y) \text{ is a vertex} \\ (\bar{a},x) \leftrightarrow (\bar{b},y)}} \chi(\bar{b})\phi(y) = \sum_{\substack{\bar{b} \in \mathbb{F}_q/H \\ y \in \mathbb{F}_q^* \\ xy \in \overline{a+b}}} \chi(\bar{b})\phi(y)$$

First, we make two changes of variables in this sum. The first change is to switch $\bar{b}$ to $\bar{c}$ by the transformation $\bar{c} = \overline{a+b} = \bar{a} + \bar{b}$.

$$\sum_{\substack{\bar{c} \in \mathbb{F}_q/H \\ y \in \mathbb{F}_q^* \\ xy \in \bar{c}}} \chi(\overline{c-a})\phi(y)$$

Next, switch y to z by the transformation $z = xy$.

$$\sum_{\substack{\bar{c}\in\mathbb{F}_q/H \\ z\in\mathbb{F}_q^* \\ z\in\bar{c}}} \chi(\overline{c-a})\phi\left(\frac{z}{x}\right).$$

Using that χ and ϕ are characters (homomorphisms), this transforms to

$$(\chi(\bar{a})\phi(x))^{-1}\sum_{\substack{\bar{c}\in\mathbb{F}_q/H \\ z\in\mathbb{F}_q^* \\ z\in\bar{c}}} \chi(\bar{c})\phi(z) = \overline{\chi(\bar{a})\phi(x)} \sum_{\left\{(\bar{c},z):\bar{c}\in\mathbb{F}_q/H, z\in\mathbb{F}_q^*, z\in\bar{c}\right\}} \chi(\bar{c})\phi(z)$$

There is an obvious bijection between the set $\left\{(\bar{c},z) : \bar{c}\in\mathbb{F}_q/H, z\in\mathbb{F}_q^*, z\in\bar{c}\right\}$ and the set $\left\{z : z\in\mathbb{F}_q^*\right\}$, since once z is picked, there is a unique coset containing z. Thus the above sum can be simplified to

$$\overline{\chi(\bar{a})\phi(x)}\sum_{z\in\mathbb{F}_q^*}\chi(\bar{z})\phi(z).$$

Define $\Gamma_{\chi,\phi} = \sum_{z\in\mathbb{F}_q^*}\chi(\bar{z})\phi(z)$ so that $\Gamma_{\chi,\phi}$ is some constant depending only on χ and ϕ. Then the vector $M\langle\chi,\phi\rangle$ is $\Gamma_{\chi,\phi}\langle\overline{\chi},\overline{\phi}\rangle$. Thus $M^2\langle\chi,\phi\rangle = \Gamma_{\chi,\phi}\overline{\Gamma_{\chi,\phi}}\langle\chi,\phi\rangle$, so $\Gamma_{\chi,\phi}\overline{\Gamma_{\chi,\phi}}$ is an eigenvalue of M^2.

Lemma 18. *Let A be a finite group. There are $|A|$ characters of A and if $\tau : A\to\mathbb{C}$ is a non-principal character then $\sum_{a\in A}\tau(a) = 0$.*

The above lemma shows there are $|\mathbb{F}_q/H|\cdot|\mathbb{F}_q^*| = q(q-1)/t = |V(G^+)|$ vectors $\langle\chi,\phi\rangle$. Secondly, the lemma shows $\langle\chi,\phi\rangle$ is orthogonal to $\langle\chi',\phi'\rangle$ if $\chi\neq\chi'$ or $\phi\neq\phi'$ (the dot product of $\langle\chi,\phi\rangle$ with $\langle\chi',\phi'\rangle$ is a sum which can be rearranged to apply Lemma 18.)

Since $\{\langle\chi,\phi\rangle : \chi,\phi \text{ characters }\}$ is a linearly independent set of $|V(G^+)|$ eigenvectors of M^2 and M^2 has $|V(G^+)|$ columns, all eigenvalues of M^2 are of the form $\Gamma_{\chi,\phi}\overline{\Gamma_{\chi,\phi}}$. The eigenvalues of M^2 are the squares of the eigenvalues of M. Since M is symmetric, these eigenvalues are real so all eigenvalues of M are of the form $\pm|\Gamma_{\chi,\phi}|$.

When χ and ϕ are principal characters of their respective groups (this means χ and ϕ map everything to 1), the corresponding eigenvalue is $q-1$ since there are $q-1$ terms in the sum defining $\Gamma_{\chi,\phi}$. This eigenvalue has multiplicity one. When χ is principal but ϕ is not principal, the eigenvalues are

$$\Gamma_{\chi,\phi} = \sum_{z\in\mathbb{F}_q^*}\phi(z) = 0.$$

There are $q-1$ possible characters ϕ, but one of them is principal so 0 will have multiplicity $q-2$ as an eigenvalue. When ϕ is principal but χ is not, we obtain

$$\Gamma_{\chi,\phi} = \sum_{z\in\mathbb{F}_q^*}\chi(\bar{z}) = t\sum_{\bar{z}\in\mathbb{F}_q/H}\chi(\bar{z}) - \chi(\bar{0}) = -\chi(\bar{0}) = -1.$$

($\chi(\bar{0})$ is subtracted since the sum over $\mathbb{F}_q^*$ will have $t = |H|$ terms for each coset, except the zero coset will only appear $t-1$ times.) Thus the eignevalues when ϕ is principal and χ is not are ± 1. For the multiplicities, there are $q/t - 1$ non-principal characters χ. They come in pairs, since if χ is a character, the complex conjugate $\overline{\chi}$ is a character as well. Also, note that $\langle \chi, \phi \rangle + \langle \bar{\chi}, \phi \rangle$ has eigenvalue 1 and $\langle \chi, \phi \rangle - \langle \bar{\chi}, \phi \rangle$ has eigenvalue -1 (when ϕ is principal.) Thus if p is an odd prime, 1 and -1 will each have multiplicity $\frac{1}{2}(q/t - 1)$.

When neither χ nor ϕ is a principal character, we apply a theorem on Gaussian sums of characters.

Theorem 19. *If χ' and ϕ are additive and multiplicative non-principal characters of $\mathbb{F}_q$ and $\mathbb{F}_q^*$ respectively, then $\left|\sum_{x \in \mathbb{F}_q^*} \chi'(x)\phi(x)\right| = \sqrt{q}$.*

While we can't apply this theorem directly since χ is not a character on $\mathbb{F}_q$, define a new additive character χ' on $\mathbb{F}_q$ as follows: for $x \in \mathbb{F}_q$ let $\chi'(x) = \chi(\bar{x})$. This is an additive character because $\chi'(0) = \chi(\bar{0}) = 1$, $\chi'(x+y) = \chi(\overline{x+y}) = \chi(\bar{x} + \bar{y}) = \chi(\bar{x})\chi(\bar{y}) = \chi'(x)\chi'(y)$, and $\chi'(-x) = \chi(\overline{-x}) = \chi(\bar{x})^{-1} = \chi'(x)^{-1}$. We can now rewrite $\Gamma_{\chi,\phi}$ as

$$\Gamma_{\chi,\phi} = \sum_{z \in \mathbb{F}_q^*} \chi'(x)\phi(x).$$

Theorem 19 shows that when χ and ϕ are both non-principal, the corresponding eigenvalue is $\pm\sqrt{q}$. $\square$

Proof of Lemma 17. Let M be the adjacency matrix of $G^\times$. Let χ be an arbitrary multiplicative character of $\mathbb{F}_q^*/H$ and let ϕ be an arbitrary additive character of $\mathbb{F}_q$.

Let $\langle \chi, \phi \rangle$ denote the column vector whose coordinates are labeled by the elements of $V(G^\times)$ and whose entry at the coordinate $(\bar{a}, x)$ is $\chi(\bar{a})\phi(x)$. We now show that $\langle \chi, \phi \rangle$ is an eigenvector of M and compute its eigenvalue. The following expression is the entry of the vector $M\langle \chi, \phi \rangle$ at the coordinate $(\bar{a}, x)$.

$$\sum_{\substack{(\bar{b},y) \text{ is a vertex} \\ (\bar{a},x) \leftrightarrow (\bar{b},y)}} \chi(\bar{b})\phi(y) = \sum_{\substack{\bar{b} \in \mathbb{F}_q^*/H \\ y \in \mathbb{F}_q \\ x+y \in \overline{ab}}} \chi(\bar{b})\phi(y)$$

First, we make two changes of variables in this sum. The first change is to switch $\bar{b}$ to $\bar{c}$ by the transformation $\bar{c} = \overline{ab} = \bar{a} \cdot \bar{b}$.

$$\sum_{\substack{\bar{c} \in \mathbb{F}_q^*/H \\ y \in \mathbb{F}_q \\ x+y \in \bar{c}}} \chi(\overline{ca^{-1}})\phi(y)$$

Next, switch y to z by the transformation $z = x + y$.

$$\sum_{\substack{\bar{c} \in \mathbb{F}_q^*/H \\ z \in \mathbb{F}_q \\ z \in \bar{c}}} \chi(\overline{ca^{-1}})\phi(z - x).$$

Using that χ and ϕ are characters (homomorphisms), this transforms to

$$(\chi(\bar{a})\phi(x))^{-1} \sum_{\substack{\bar{c}\in\mathbb{F}_q^*/H \\ z\in\mathbb{F}_q \\ z\in\bar{c}}} \chi(\bar{c})\phi(z) = \overline{\chi(\bar{a})\phi(x)} \sum_{\left\{(\bar{c},z):\bar{c}\in\mathbb{F}_q^*/H, z\in\mathbb{F}_q, z\in\bar{c}\right\}} \chi(\bar{c})\phi(z)$$

There is an obvious bijection between the set $\left\{(\bar{c}, z) : \bar{c} \in \mathbb{F}_q^*/H, z \in \mathbb{F}_q, z \in \bar{c}\right\}$ and the set $\left\{z : z \in \mathbb{F}_q^*\right\}$, since once a non-zero z is picked, there is a unique coset containing z. (When $z = 0$, there is no coset containing z.) Thus the above sum can be simplified to

$$\overline{\chi(\bar{a})\phi(x)} \sum_{z\in\mathbb{F}_q^*} \chi(\bar{z})\phi(z).$$

Define $\Gamma_{\chi,\phi} = \sum_{z\in\mathbb{F}_q^*} \chi(\bar{z})\phi(z)$ so that $\Gamma_{\chi,\phi}$ is some constant depending only on χ and ϕ. Then the vector $M\langle\chi,\phi\rangle$ is $\Gamma_{\chi,\phi}\left\langle\overline{\chi},\overline{\phi}\right\rangle$. Thus $M^2\langle\chi,\phi\rangle = \Gamma_{\chi,\phi}\overline{\Gamma_{\chi,\phi}}\langle\chi,\phi\rangle$ so $\Gamma_{\chi,\phi}\overline{\Gamma_{\chi,\phi}}$ is an eigenvalue of M^2. Like the last proof, Lemma 18 shows all eigenvalues of M^2 are of the form $\Gamma_{\chi,\phi}\overline{\Gamma_{\chi,\phi}}$ so all eigenvalues of M are of the form $\pm|\Gamma_{\chi,\phi}|$.

When χ and ϕ are principal characters of their respective groups, the corresponding eigenvalue is $q-1$ since there are $q-1$ terms in the sum. This eigenvalue has multiplicity one. When ϕ is principal but χ is not principal, the eigenvalues are

$$\Gamma_{\chi,\phi} = \sum_{z\in\mathbb{F}_q^*} \chi(\bar{z}) = t \sum_{\bar{z}\in\mathbb{F}_q^*/H} \chi(\bar{z}) = 0.$$

There are $(q-1)/t$ possible characters χ, but one of them is principal so 0 will have multiplicity $(q-1)/t-1$ as an eigenvalue. When χ is principal but ϕ is not, we obtain

$$\Gamma_{\chi,\phi} = \sum_{z\in\mathbb{F}_q^*} \phi(z) = \sum_{z\in\mathbb{F}_q} \phi(z) - \phi(0) = -\phi(0) = -1.$$

Thus the eignevalues when χ is principal and ϕ is not are ± 1. For the multiplicities, there are $q-1$ non-principal characters ϕ. They come in pairs, since if ϕ is a character, the complex conjugate $\overline{\phi}$ is a character as well. Also, note that $\langle\chi,\phi\rangle + \left\langle\chi,\bar{\phi}\right\rangle$ has eigenvalue 1 and $\langle\chi,\phi\rangle - \left\langle\chi,\bar{\phi}\right\rangle$ has eigenvalue -1 (when χ is principal.) Thus if p is an odd prime, 1 and -1 will each have multiplicity $\frac{1}{2}(q-1)$.

When neither χ or ϕ is a principal character, we apply Theorem 19. While we can't apply this theorem directly since χ is not a multiplicative character on $\mathbb{F}_q^*$, define a new multiplicatve character χ' on $\mathbb{F}_q^*$ as follows: for $x \in \mathbb{F}_q^*$ let $\chi'(x) = \chi(\bar{x})$. This is a multiplicative character because $\chi'(1) = \chi(\bar{1}) = 1$, $\chi'(xy) = \chi(\overline{xy}) = \chi(\bar{x}\cdot\bar{y}) = \chi(\bar{x})\chi(\bar{y}) = \chi'(x)\chi'(y)$, and $\chi'(x^{-1}) = \chi(\overline{x^{-1}}) = \chi(\bar{x})^{-1} = \chi'(x)^{-1}$. We can now rewrite $\Gamma_{\chi,\phi}$ as

$$\Gamma_{\chi,\phi} = \sum_{z\in\mathbb{F}_q^*} \chi'(x)\phi(x).$$

Theorem 19 shows that when χ and ϕ are both non-principal, the corresponding eigenvalue is $\pm\sqrt{q}$. $\square$

4.3 Independence number

In Table 2, there is no lower bound in the upper right cell; that is, when m is much larger than t the only lower bound we know is the bound of $c_t m^{2-1/t}$ from the random graph. What about using G^+ or $G^\times$ as the first color in a construction for the lower bound? In other words, what is the independence number of G^+ and $G^\times$? This is related to the conjecture that Paley Graphs are Ramsey Graphs (see [17] and its references.) While we aren't able to determine exactly the independence number, computation suggests that G^+ and $G^\times$ have independent sets of size roughly $\sqrt{n}$, where n is the number of vertices. In particular, computation suggests the following conjecture for G^+.

Conjecture 20. *Let $G^+(q,t)$ be the graph constructed at the beginning of this section for the parameters q and t. Recall that $G^+(q,t)$ has $q(q-1)/t$ vertices which is regular of degree $q-1$ so $G^+(2^a, 2^{a-1})$ is an n-vertex graph where every degree is about $n/2$ and any pair of vertices have about $n/4$ common neighbors. For $a \geq 6$,*

$$\alpha(G^+(2^a, 2^{a-1})) = \begin{cases} 2^{a/2} & \textit{if } a \textit{ is even} \\ 2^{(a-1)/2}+1 & \textit{if } a \textit{ is odd} \end{cases}$$

$$\alpha(G^+(p^2, p)) = p^2 - 1 \qquad \textit{if } p \textit{ is odd}$$

Note that $\alpha(G^+(2^3, 2^2)) = 4$ and $\alpha(G^+(2^4, 2^3)) = 5$, which don't quite match the conjecture. For $\alpha(G^+(2^a, 2^{a-1})$, the conjecture is true for $a = 6, 7, 8, 9, 10$. For $G^+(p^2, p)$, the conjectured value is $p^2 - 1$; we can prove a lower bound of $\frac{1}{2}p^2$. First, we need the following simple lemma about finite fields and field extensions.

Lemma 21. *Let p be a prime and let $x \in \mathbb{F}_{p^a}^*$ with x a generator for the cyclic multiplicative group $\mathbb{F}_{p^a}^*$. Then*

$$\{1, 2, \ldots, p-1\} = \left\{x^{t(p^a-1)/(p-1)} : 0 \leq t < p-1\right\}$$

Proof. The Frobenius automorphism $\phi(z) = z^p$ has fixed points exactly the elements in $\mathbb{Z}_p$. Thus

$$\phi(x^{t(p^a-1)/(p-1)}) = x^{tp(p^a-1)/(p-1)} = x^{t(p^a-1)} x^{t(p^a-1)/(p-1)}.$$

Since $x^{q-1} = 1$, $x^{t(p^a-1)/(p-1)}$ is a fixed point so it is in $\mathbb{Z}_p$. Also, since the multiplicative group of $\mathbb{F}_q$ is cyclic, the elements $x^{t(p^a-1)/(p-1)}$ are distinct and there are $p-1$ of them. $\square$

Lemma 22. *If p is an odd prime, then $\alpha(G^+(p^2, p)) \geq \lfloor p^2/2 \rfloor$.*

Proof. $q = p^2$, $t = p$, so $n = p^2(p-1)$. Thus $\frac{1}{2}n^{2/3} \leq \frac{1}{2}p^2 = \frac{1}{2}q$.

The field $\mathbb{F}_q$ is $\mathbb{Z}_p[x]/(f(x))$, where $f(x)$ is some irreducible polynomial of degree 2. Thus elements of $\mathbb{F}_q$ can be written as $\alpha x + \beta$ for $\alpha, \beta \in \mathbb{Z}_p$. Since $t = p$, we need H to be an additive subgroup of $\mathbb{F}_q$ of order p. The additive subgroup generated by x has order p, so let $H = \{0, x, 2x, 3x, \ldots, (p-1)x\}$. We now claim the following set is an independent set:

$$\left\{(\bar{0}, x^{2k}) : 0 \leq k < q/2\right\}.$$

Consider two vertices in this set: $(\bar{0}, x^{2j})$ and $(\bar{0}, x^{2k})$. These will be adjacent if $x^{2j+2k} \in \bar{0} = H$, in other words $x^{2j+2k-1} \in \mathbb{Z}_p$. But from Lemma 21, the powers of x which give elements in $\mathbb{Z}_p$ are of the form $t(p+1)$ for some t. Since p is an odd prime, $p+1$ is even. Thus $x^{2j+2k-1} \notin \mathbb{Z}_p$. $\square$

Most likely, the above proof can be extended to $G^+(p^a, p^b)$ when b divides a as follows. Let $q = p^a$ and view the field $\mathbb{F}_q$ as an extension field over $\mathbb{F}_p$; the Galois group $Gal(\mathbb{F}_q/\mathbb{F}_p)$ is cyclic of order a with generator the Frobenius automorphism used in Lemma 21. Since b divides a, there is a subgroup of $Gal(\mathbb{F}_q/\mathbb{F}_p)$ of order a/b. By the fundamental theorem of Galois theory, this corresponds to an intermediate field extension of order p^b. Thus we have a subfield of $\mathbb{F}_q$ of order p^b and an automorphism ϕ which fixes this subfield. Replace the Frobenius automorphism in the above proof by this ϕ, investigate which powers of x are fixed by ϕ, and find a set whose sums avoid these powers of x to construct an independent set in $G^+(p^a, p^b)$.

5 An upper bound on $r(K_{2,t}, K_3)$

Proof of Theorem 6. The lower bound in Theorem 6 is a slightly weaker statement than Proposition 2. Therefore, we turn our attention to proving the upper bound. Consider a coloring of $E(K_n)$ which avoids $K_{2,t}$ in the first color and K_3 in the second color. Let G be the graph consisting of the edges in the first color and H be the graph consisting of edges in the second color. Let X be a maximum independent set in H, let $Y = V(H) - X$. Define two reals a and d by $dn = |X|$ and $an^2 = |E(G[X,Y])|$, where $G[X,Y]$ is the graph comprising edges with one endpoint in X and the other endpoint in Y.

Lemma 23. $d(1-2d) \le a \le d(1-d)$

Proof. There are at most $d(1-d)n^2$ cross edges in $G \cup H$, and of these, an^2 is the number of edges in G Thus $a \le d(1-d)$.

Since H is K_3-free, the maximum degree in H is at most the independence number of H. Thus since X is a maximum independent set of size dn, the maximum degree in H is at most dn. There are an^2 XY-edges in G so $d(1-d)n^2 - an^2$ XY-edges in H. Consider the average cross-H-degree of a vertex in X, that is, the quantity

$$\frac{1}{|X|} \sum_{x \in X} |N_H(x) \cap Y| = \frac{d(1-d)n^2 - an^2}{dn}.$$

This is at most the maximum degree of H which is dn. Therefore,

$$\begin{aligned} \frac{(d(1-d)-a)n^2}{dn} &\le dn, \\ d(1-d) - a &\le d^2, \\ d(1-2d) = d(1-d) - d^2 &\le a. \end{aligned}$$

$\square$

Lemma 24.

$$\frac{n}{t} < (1+o(1))\frac{d^2}{d^3 + \frac{a^2}{1-d}}$$

Proof. Count paths of length two in G where the endpoints are in X. There are at most $t\binom{dn}{2}$ such paths since G is $K_{2,t+1}$-free. There are two cases: all three vertices are in X or the middle vertex is in Y. $G[X]$ is a complete graph, so counting paths of length two in $G[X]$, there are dn vertices for the center and $\binom{dn-1}{2}$ choices for the endpoints. For XYX paths, there are $(1-d)n$ choices for the center $y \in Y$, and then $\binom{d_X(y)}{2}$ choices for the endpoints, where $d_X(y) = |N_G(y) \cap X|$. By Cauchy-Schwartz,

$$\begin{aligned}\sum_{y\in Y}\binom{d_X(y)}{2} &\geq (1-o(1))\frac{1}{2}|Y|\left(\frac{\sum_{y\in Y} d_X(y)}{|Y|}\right)^2 \\ &\geq (1-o(1))\frac{1}{2}(1-d)n\left(\frac{an^2}{(1-d)n}\right)^2 \\ &= (1-o(1))\frac{a^2n^3}{2(1-d)}\end{aligned}$$

Since there are at most $t\binom{dn}{2}$ paths with both endpoints in X and there are $dn\binom{dn-1}{2}$ XXX paths,

$$\begin{aligned}dn\binom{dn-1}{2} + \frac{a^2n^3}{2(1-d)} &< (1+o(1))t\binom{dn}{2} \\ \frac{1}{2}d^3n^3 + \frac{a^2n^3}{2(1-d)} &< (1+o(1))\frac{1}{2}td^2n^2 \\ n\left(d^3 + \frac{a^2}{1-d}\right) &< (1+o(1))td^2\end{aligned}$$

$\square$

The upper bound in Lemma 24 is worst when a is as small as possible. By Lemma 23, $a \geq d(1-2d)$ so

$$\frac{n}{t} < (1+o(1))\frac{d^2}{d^3 + \frac{d^2(1-2d)^2}{1-d}}.$$

The left hand side is maximized when $d = 1 - \frac{\sqrt{3}}{3} \approx 0.42$ with a value of $\frac{2-\sqrt{3}}{12-7\sqrt{3}} \approx 2.155$, completing the proof of Theorem 6. $\square$

6 Conclusion and open problems

- Can improvements be made on $r(K_{2,t}, K_3)$? We conjecture that $r(K_{2,t}, K_3) = 2t + O(1)$. Analyzing the bound in Lemma 24, the range where it is bigger than two is when d is between $1/3$ and $1/2$, so only when d is in this range must a better bound be found. Lemma 24 was obtained by counting paths of length two of a certain type. We investigated counting other types of paths in an attempt to find an upper bound which would be better in the range $1/3 \leq d \leq 1/2$, but all our attempts gave worse bounds. When $d = 1/3$ for example, we can't seem to find anything to count which will improve upon the count from Lemma 24.

- The upper bound from Theorem 6 in Section 5 can be extended easily to more colors. Let $G_1, \ldots, G_k$ be the $K_{2,t}$-free colors and H be the K_3-free color. All the arguments in Section 5 worked by counting things in G by average degree and then using Cauchy-Schwartz. Using one more application of Cauchy-Schwartz, these averaging arguments can be extended to $G_1, \ldots, G_k$.

- Looking at Table 2, it is somewhat strange that when m is around $\log^2 t$ the best lower bound switches from a simple construction (the Túran Graph) to the random graph. Perhaps some combination of these two constructions could provide a good lower bound when m is around $\log^2 t$. Unfortunately, the two simple ideas do not work. One option is to take ℓ random graphs forbidding $K_{2,t}$ and independence number m/ℓ as one color and all edges between the random graphs as the second color. Another option is to take ℓ cliques in red (of some size smaller than $t+1$) and put a random graph between cliques. We are unable to make either of these two constructions beat the bounds in Table 2, even for a restricted range of m.

- The ideas in this paper can be extended to $r_k(K_{s,t}; K_m)$ when s is fixed using field norms, similar to the projective norm graphs. Let $N : \mathbb{F}_{q^s} \to \mathbb{F}_q$ be the field norm of the extension of $\mathbb{F}_{q^s}$ over $\mathbb{F}_q$. (When q is prime $N(x) = x^{(q^s-1)/(q-1)}$ and when q is a prime-power the field norm is more complicated.) Given q, t, and s, let H be an additive subgroup of $\mathbb{F}_q$ of order t and form a graph G^+ as follows. The vertex set is $(\mathbb{F}_q/H) \times \mathbb{F}_{q^s}^*$ and two vertices $(\bar{a}, x)$ and $(\bar{b}, y)$ are adjacent if $N(xy) \in \overline{a+b}$. The graph $G^\times$ can be similarly extended using norms. These constructions will now avoid $K_{s,t}$ when $t \geq (s-1)! + 1$. Using ideas from [20], the computations in Section 4.2 can be extended to find the spectrum of G^+ and $G^\times$. Theorem 11 can then be used to prove a lower bound on $r_k(K_{s,t}; K_m)$ when $k \geq 2$ and s is fixed.

  Extending to s not fixed, somehow the vertex set should be taken as $(\mathbb{F}_q/H) \times \left(\mathbb{F}_q^*/J\right)$ where H is an additive subgroup of order t and J is a multiplicative subgroup of order s. But if $(\bar{a}, \bar{x})$ and $(\bar{b}, \bar{y})$ are vertices, it is not clear what the relation between $\overline{a+b}$ and $\overline{xy}$ is. Perhaps if H and J are taken not as arbitrary subgroups but specific subgroups with certain properties, the coset $\overline{a+b}$ would be completely contained in or disjoint from $\overline{xy}$.

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A Density of the Prime Numbers

In this appendix, we prove Lemma 12. For convenience, we restate the lemma here.

Lemma 12. *Fix integers $s, L \geq 1$. There exists a constant $\delta > 0$ depending only on s and L such that the following holds. For every $t \geq 2$ and $m \geq 4^s L \log^s t$, either $\frac{\delta m^2 t}{L^2 \log^{2s}(mt)} \leq 2$ or there is a prime power q so that $q \equiv 1 \pmod{t}$ and*

$$\delta \frac{m^2 t}{L^2 \log^{2s}(mt)} \leq \frac{q(q-1)}{t} \leq \frac{m^2 t}{L^2 \log^{2s}(mt)}.$$

Dirichlet's Theorem states that if $gcd(t, a) = 1$ then there are infinitely many prime numbers p with $p \equiv a \pmod{t}$ so there are infinitely many prime numbers congruent to one modulo t. This isn't quite enough for us since we need to find a prime in a specific range, but the prime number theorem for arithmetic progressions states more than Dirichlet's theorem; essentially it says that the primes are asymptotically equally divided modulo t into the $\phi(t)$ congruence classes coprime to t, where $\phi(t)$ is the Euler totient function.

Theorem 25. *(Prime Number Theorem in Arithmetic Progressions) Let $\pi(x; t, a)$ be the number of primes less than or equal to x and congruent to a modulo t. Then*

$$\pi(x; t, a) = (1 + o_t(1)) \frac{1}{\phi(t)} \frac{x}{\log x}.$$

The subscript of t on o implies the constant in the definition of o can depend only on t. In particular, when t gets big there are primes congruent to 1 $\pmod{t}$ between $(\ell - 0.01)t$ and ℓt.

Corollary 26. *There exists an absolute constant T_0 so that if $t \geq T_0$ and $\ell > 1.01$, then there exists a prime congruent to one modulo t between ℓt and $(\ell - 0.001)t$.*

Note that both Theorem 25 and Corollary 26 are not the best known results of this kind, but are (more than) enough for our purposes. For example, the requirement that $\ell > 1.01$ in Corollary 26 is an easy way to overcome the fact that $\phi(t)$ can be as large as $t-1$ and to have at least one prime, $x/\log x$ from Theorem 25 must be at least $(1+o(1))\phi(t)$. Requiring $x \geq 1.01t$ and $t \geq T_0$ easily implies $x \gg \phi(t)$.

Before the proof of Lemma 12, we get some computations out of the way.

Lemma 27. *If $m, L, s, t \geq 1$ are real numbers, then there exists a constant M_0 depending only on s and L so that if $m \geq M_0$ and $m \geq 4^s L \log^s t$, then*

$$\frac{m}{L \log^s(mt)} > 1.01.$$

Proof. Pick M_0 large enough so that for $m \geq M_0$, $2^s \log^s m < \frac{m}{1000L}$. Then

$$\begin{aligned}\log^s(mt) &= (\log m + \log t)^s \leq (2 \max\{\log m, \log t\})^s = 2^s \max\{\log^s m, \log^s t\} \\ &\leq 2^s \log^s m + 2^s \log^s t < \frac{m}{1000L} + \frac{m}{2^s L} \leq \frac{m}{1000L} + \frac{m}{2L} \leq \frac{m}{1.01L}.\end{aligned}$$

Therefore,

$$\frac{m}{L \log^s(mt)} > \frac{m}{L(m/1.01L)} = 1.01.$$

$\square$

Proof of Lemma 12. For notational convenience, define $\ell = \frac{m}{L \log^s(mt)}$. To prove the lemma, we must produce a $\delta > 0$ so that for any $t \geq 2$ and $m \geq 4^s L \log^s t$, either $\delta \ell^2 t \leq 2$ or there exists a prime power q so that $q \equiv 1 \pmod t$ and $\delta \ell^2 t \leq q(q-1)/t \leq \ell^2 t$.

Let T_0 and M_0 be the constants from Corollary 26 and Lemma 27 respectively, and define T_1 so that $M_0 = 4^s L \log^s T_1$. The constants T_0, T_1, and M_0 depend only on s and L. Define δ small enough so that the following equations are satisfied:

$$\frac{\delta M_0^2 T_1}{L^2 \log^{2s}(M_0 T_1)} \leq 2, \quad \delta (1.01 T_0)^2 T_0 \leq 2, \quad \delta < \frac{1}{16}.$$

The definition of δ depends only on s and L as required.

Assume that $\delta \ell^2 t > 2$. We must now find a prime power q so that $q \equiv 1 \pmod t$ and $\delta \ell^2 t \leq q(q-1)/t \leq \ell^2 t$. Multiplying everything by t and taking the square root, we must find q between

$$\sqrt{\delta} \ell t \leq \sqrt{q(q-1)} \leq \ell t. \tag{16}$$

$\sqrt{q(q-1)}$ is approximately q; in fact, if we can find q in the following range

$$2\sqrt{\delta}\ell t \leq q \leq \ell t, \tag{17}$$

then (16) will be satisfied. This is because

$$\sqrt{q(q-1)} = \sqrt{q}\sqrt{q-1} \geq \sqrt{q} \cdot \frac{\sqrt{q}}{2} = \frac{q}{2},$$

so if we find $q \geq 2\sqrt{\delta}\ell t$, then $\sqrt{q(q-1)} \geq q/2 \geq \sqrt{\delta}\ell t$ so that (16) is satisfied.

We now divide into cases depending on if $t \geq T_0$ or $m \geq M_0$.

- **Case 1:** $m \geq M_0$ **and** $t \geq T_0$: Lemma 27 shows $\ell > 1.01$ and Corollary 26 then shows there is a prime q congruent to one modulo t between $(\ell - 0.001)t$ and ℓt. Since $\delta < \frac{1}{16}$, $2\sqrt{\delta}\ell < \ell - 0.001$. We have now found q in the range from (17).

- **Case 2:** $m < M_0$: By assumption, $m \geq 4^s L \log^s t$. Thus $m < M_0$ and the definition of T_1 shows that $t \leq T_1$. But then,

$$\delta \ell^2 t \leq \frac{\delta M_0^2 T_1}{L^2 \log^{2s}(M_0 T_1)} \leq 2$$

by the definition of δ, and this contradicts that $\delta \ell^2 t > 2$.

- **Case 3:** $m \geq M_0$ **and** $t < T_0$ **and** $\ell / T_0 > 1.01$: Let $t' = tT_0$ so $t' \geq T_0$ and $\ell' = \ell / T_0 > 1.01$. Corollary 26 show that there exists a prime q congruent to one modulo t' between $(\ell' - 0.001)t'$ and $\ell' t'$. That is,

$$\left(\frac{\ell}{T_0} - 0.001\right) tT_0 \leq q \leq \frac{\ell}{T_0} \cdot tT_0 = \ell t.$$

We now want to show that q is in the range (17). In other words, show

$$2\sqrt{\delta}\ell < \left(\frac{\ell}{T_0} - 0.001\right) T_0$$
$$2\sqrt{\delta} \cdot \frac{\ell}{T_0} < \frac{\ell}{T_0} - 0.001.$$

Written this way, we can easily see that since $\delta < 1/16$, this inequality is true since $\ell / T_0 > 1.01$. Lastly, q congruent to one modulo $t' = tT_0$ implies q is congruent to one modulo t, so we have found q with the required properties.

- **Case 4:** $t < T_0$ **and** $\ell / T_0 < 1.01$: In this case, $t < T_0$ and $\ell < 1.01 T_0$ implies

$$\delta \ell^2 t \leq \delta (1.01 T_0)^2 T_0 \leq 2$$

by the definition of δ, but this contradicts that $\delta \ell^2 t > 2$.

□

B Lower bounds on $r_k(K_{2,t}; K_m)$ for $k \geq 3$

In this appendix, we sketch the proof that inequality (2) in Theorem 11 is true when $d = \sqrt{nt}$, $\lambda = (nt)^{1/4}$, and $m = 2k\sqrt{n/t}\log n$. In the computations to follow, let $\theta = \sqrt{n/t}\log n$ which will simplify the notation. The inequality (2) is (temporarily disregard the constants)

$$\left(\frac{md^2}{\lambda n \log n}\right)^{\frac{2kn\log n}{d}} \left(\frac{\lambda n}{md}\right)^{km} \left(\frac{m}{n}\right)^{m(k-1)} < 1.$$

Substituting $d = \sqrt{nt}$ and $\lambda = (nt)^{1/4}$, this simplifies to

$$\left(\frac{mnt}{(nt)^{1/4} n \log n}\right)^{\frac{2k\sqrt{n}\log n}{\sqrt{t}}} \left(\frac{(nt)^{1/4} n}{m\sqrt{nt}}\right)^{km} \left(\frac{m}{n}\right)^{m(k-1)} < 1.$$

Simplifying, this is

$$\left(\frac{mt^{3/4}}{n^{1/4}\log n}\right)^{2k\theta} \left(\frac{n^{3/4}}{mt^{1/4}}\right)^{km} \left(\frac{m}{n}\right)^{m(k-1)} < 1.$$

Substitute in $m = 2k\theta$:

$$\left(\frac{\theta t^{3/4}}{n^{1/4}\log n}\right)^{2k\theta} \left(\frac{n^{3/4}}{\theta t^{1/4}}\right)^{2k^2\theta} \left(\frac{\theta}{n}\right)^{2k\theta(k-1)} < 1.$$

Drop a $2k\theta$ in the exponent, and substitute in $\theta = \sqrt{n/t}\log n$:

$$\left(n^{1/4}t^{1/4}\right) \left(\frac{n^{1/4}t^{1/4}}{\log n}\right)^{k} \left(\frac{\log n}{n^{1/2}t^{1/2}}\right)^{(k-1)} < 1.$$

Simplify to

$$(nt)^{\frac{1}{4}+\frac{k}{4}-\frac{k-1}{2}} \log^{-1} n < 1.$$

When $k \geq 3$, the exponent on nt is non-positive so the expression is true (even when we add back in the constants that got dropped.)

Thus we can conclude that for $k \geq 3$ and $m = 2k\theta = 2k\sqrt{n/t}\log n$, $r_k(K_{2,t}; K_m) > n$. Solving for n in terms of m we obtain $r_k(K_{2,t}; K_m) = \Omega(m^2 t/\log^2(mt))$, proving Theorem 4 *(ii)*.